D gr Structur and Th ir Finit Sub tructur

Ste en Lempp

University of Wisconsin-Madison

November 8, 2024

We work with subsets of **N** usually enote as *!*) an stu y their relative computational complexity.

Definition

- *redu ibility* is a transitive reflexive relation $_r$ on $P(!)$ so that A $_r$ B expresses that B "can compute" A).

We work with subsets of **N** usually enote as $!$) an stu y their relative computational complexity.

Definition

- *redu ibility* is a transitive reflexive relation $_r$ on $P(!)$ so that A $_r$ B expresses that B “can compute” A).
- $A; B$ $!$ are *r-equivalent* written A $_r$ B) if A $_r$ B an B $_r$ A. A an B have “equal computational content”.)

We work with subsets of $\mathbf{N}$

We work with subsets of $\mathbf{N}$ usually enote as !) an stu y their relative computational complexity.

Definition

[redacted]

[redacted]

Definitions and Examples

[redacted]

Many re ucibilities have been consi ere in classical computability theory, theoretical computer science an even set theory:

- $A \quad _m B$ if there is a computable function f such that $x\ 2\ A$ i $f(x)\ 2\ B$.
- $A \quad _T B$ if there is a Turing functional Φ with $A = \Phi(B)$.
- $A \quad _e B$ if there is an enumeration operator Φ with $A = \Phi(B)$.
- $A \quad ^p_m B$ if $A \quad _m B$ via a polynomial-time function f.
- $A \quad ^p_T B$ if $A \quad _T B$ via a polynomial-time functional Φ.

[redacted]

Many re ucibilities have been consi ere in classical computability theory, theoretical computer science an even set theory:

- $A \ _m B$ if there is a computable function f such that x 2 A i $f(x)$ 2 B.
- $A \ _T B$ if there is a Turing functional Φ with $A = \Phi(B)$.
- $A \ _e B$ if there is an enumeration operator Φ with $A = \Phi(B)$.
- $A \ ^p_m B$ if $A \ _m B$ via a polynomial-time function f.
-

Degree theory stu ies egree structures as algebraic objects, i.e., as partial or ers, sometimes in a richer language.
For most "natural" egree structures D, we have:

- D has a least element $\mathbf{0}_D$.

Degree theory stu ies egree structures as algebraic objects, i.e., as partial or ers, sometimes in a richer language.
For most “natural” egree structures D, we have:

- D has a least element $\mathbf{0}_D$.
- Local egree structures have a greatest element, global egree structures o not.
- D is *lo ally ountable*, i.e., any egree has at most countably many pre ecessors.

Degree theory stu ies egree structures as algebraic objects, i.e., as partial or ers, sometimes in a richer language.
For most "natural" egree structures D, we have:

- D has a least element $\mathbf{0}_D$.
- Local egree structures have a greatest element, global egree structures o not.
- D is *lo ally ountable*, i.e., any egree has at most countably many pre ecessors.
- D is an upper semilattice but usually not a lattice), i.e., D has a join operation $\deg(A)$ [$\deg(B) = \deg(A \quad B)$, where $A \quad B = f2x \; j \; x \; 2 \; Ag$ [$f2x + 1 \; j \; x \; 2 \; Bg$.

Degree theory stu ies egree structures as algebraic objects, i.e., as partial or ers, sometimes in a richer language.

For most "natural" egree structures D, we have:

- D has a least element $\mathbf{0}_D$.
- Local egree structures have a greatest element, global egree structures o not.
- D is *lo ally ountable*, i.e., any egree has at most countably many pre ecessors.
- D is an upper semilattice but usually not a lattice), i.e., D has a join operation deg(A) [deg(B) = deg(A B), where A B = f2x j x 2 Ag [f2x + 1 j x 2 Bg.
- Most global egree structures support a "jump" operation **a** 7! **a**^{0} such that **a** < **a**^{0}, an **a** implies **a**^{0} ^{0}.

“Natural” egree structures

“Natural” egree structures D ten to be very complicate an usually follow this pattern:

- The first-or er theory of the partial or er D is un eci able;

"Natural" egree structures D ten to be very complicate an usually follow this pattern:

- The first-or er theory of the partial or er D is un eci able; in fact, it usually is as complicate as secon -or er arithmetic for global egree structures) or first-or er arithmetic for countable local egree structures).

"Natural" egree structures D ten to be very complicate an usually follow this pattern:

- The first-or er theory of the partial or er D is un eci able;

"Natural" egree structures D ten to be very complicate an usually follow this pattern:

- The first-or er theory of the partial or er D is un eci able; in fact, it usually is as complicate as secon -or er arithmetic for global egree structures) or first-or er arithmetic for countable local egree structures).

Therefore, computability theorists often stu y "fragments" of the first-or er theory, etermine by a boun on the quantifier epth of the formulas:

- The 9-theory of D is eci able since all finite partial or ers embe into D).

"Natural" egree structures D ten to be very complicate an usually follow this pattern:

-

"Natural" egree structures D ten to be very complicate an usually follow this pattern:

- The first-or er theory of the partial or er D is un eci able; in fact, it usually is as complicate as secon -or er arithmetic for global egree structures) or first-or er arithmetic for countable local egree structures).

Therefore, computability theorists often stu y "fragments" of the first-or er theory, [illegible]

egree structure	complexity: 1st or 2n or er arithmetic	*9*- or *89*- fragment eci able	*989*- fragment un eci able
D_m	2n : Nero e, Shore 1980	*89*: Dëgtev 1979	Nies 1996
$D_m(\quad \mathbf{0}_m')$	1st: Nies 1994		

egree structure	complexity: 1st or 2n or er arithmetic	*9*- or *89*- fragment eci able	*989*- fragment un eci able
D_m	2n : Nero e, Shore 1980	*89*: Dëgtev 1979	Nies 1996
$D_m(\quad \mathbf{0}'_m)$	1st: Nies 1994		
D_T	2n : Simpson 1977	*89*: Lerman/ Shore 1978	Lerman, Schmerl 1983
$D_T(\quad \mathbf{0}'_T)$	1st: Shore 1981	*89*: Lerman, Shore 1988	

egree structure	complexity: 1st or 2n or er arithmetic	*9*- or *89*- fragment eci able	*989*- fragment un eci able
D_m	2n : Nero e, Shore 1980	*89*: Dëgtev 1979	Nies 1996
$D_m(\quad \mathbf{0}'_m)$	1st: Nies 1994		
D_T	2n : Simpson 1977	*89*: Lerman/ Shore 1978	Lerman, Schmerl 1983
$D_T(\quad \mathbf{0}'_T)$	1st: Shore 1981	*89*: Lerman, Shore 1988	
$D_T(\mathrm{c.e.})$	1st: Harrington, Slaman 1984	*9*: Sacks 1963	Lempp, Nies, Slaman 1998

egree structure	complexity: 1st or 2n or er arithmetic	9- or 89- fragment eci able	989- fragment un eci able
D_m	2n : Nero e, Shore 1980	89: Dëgtev 1979	Nies 1996
$D_m(\ \mathbf{0}^{0}_{m})$	1st: Nies 1994		
D_T	2n : Simpson 1977	89: Lerman/ Shore 1978	Lerman, Schmerl 1983
$D_T(\ \mathbf{0}^{0}_{T})$	1st: Shore 1981	89: Lerman, Shore 1988	
D_T(c:e:)	1st: Harrington, Slaman 1984	9: Sacks 1963	Lempp, Nies, Slaman 1998
D_e	2n : Slaman, Woo in 1997	9: Lagemann 1972	Kent 2006
$D_e(\ \mathbf{0}^{0}_{e})$	1st: Ganchev, M. Soskova 2012		

The un eci ability of the *989*-theory is usually prove using the

Nies Transfer Lemma 1996 special case)

If a class C of finite structures is *9*- efinable with parameters in a egree structure D, an the common *898*-theory of C is here itarily un eci able, then the *989*-theory of D is un eci able.

The un eci ability of the *989*-theory is usually prove using the

Nies Transfer Lemma 1996 special case)

If a class C of finite structures is *9*- efinable with parameters in a egree structure D, an the common *898*-theory of C is here itarily un eci able, then the *989*-theory of D is un eci able.

The class C use in the results cite above is

- the class of all finite istributive lattices co e as initial segments for the m- egrees, the c.e. m- egrees, an the Turing egrees;

989-Theory

The un eci ability of the *989*-theory is usually prove using the

Nies Transfer Lemma 1996 special case)

If a class *C* of finite structures is *9*- efinable with parameters in a egree structure *D*, an the common *898*-theory of *C* is here itarily un eci able, then the *989*-theory of *D* is un eci able.

The class *C* use in the results cite above is

- the class of all finite istributive lattices co e as initial segments for the *m*- egrees, the c.e. *m*- egrees, an the Turing egrees; an
- the class of all finite bipartite graphs without equality with nonempty left an right omain in elicate co ing arguments for the c.e. Turing egrees, for the enumeration egrees an for the $^0_2$-enumeration egrees.

For the enumeration egrees, one can also co e all finite istributive lattices as intervals Lempp, Slaman, M. Soskova 2021).

Deci ing the *89*-theory of D amounts to giving a uniform ecision proce ure to the following

Problem for eci ing the *89*-theory of D)

Given finite partial or ers P an Q_i P for $i < n$), oes every embe ing of P into D exten to an embe ing of Q_i into D for some $i < n$ where i may epen on the embe ing of P)?

Deci ing the *89*-theory of *D* amounts to giving a uniform ecision proce ure to the following

Problem for eci ing the *89*-theory of *D*)

Given finite partial or ers *P* an Q_i *P* for $i < n$), oes every embe ing of *P* into *D* exten to an embe ing of Q_i into *D* for some $i < n$ where *i* may epen on the embe ing of *P*)?50.9rg-4.00005

vhe[()TJ/F9 f9 Td[(ma)Bep)(nd)in)[G)g(28q)b)TJ/F)[(Theo)f)(emb)g(ng)TJETd[(A)T)(r9

Deci ing the 89-theory of D amounts to giving a uniform ecision proce ure to the following

Problem for eci ing the 89-theory of D)

Given finite partial or ers P an Q_i P for $i < n$), oes every embe ing of P into D exten to an embe ing of Q_i into D for some $i < n$ where i may epen on the embe ing of P)?

For the m- egrees an the c.e. m- egrees, one exten s P minimally to a finite istributive lattice L an embe s it into D as an initial segment; now an embe ing of L can be exten e to an embe ing of a finite partial or er Q_i L i no element of Q_i is below any element of L, an Q_i respects joins in L.

For the Turing egrees, one procee s similarly but with a finite lattice L minimally exten ing P.

Deci ing the 89-theory of D amounts to giving a uniform ecision proce ure to the following

Problem for eci ing the 89-theory of D)

Given finite partial or ers P an Q_i P for $i < n$), oes every embe ing of P into D exten to an embe ing of Q_i into D for some $i < n$ where i may epen on the embe ing of P)?

For the m- egrees an the c.e. m- egrees, one exten s P minimally to a finite istributive lattice L an embe s it into D as an initial segment; now an embe ing of L can be exten e to an embe ing of a finite partial or er Q_i L i no element of Q_i is below any element of L, an Q_i respects joins in L.
For the Turing egrees, one procee s similarly but with a finite lattice L minimally exten ing P.
For the $^0_2$-Turing egrees, embe L both as an initial segment; an also L $f1g$ as an initial segment, mapping 1 to $\mathbf{0}'_T$.

Two natural subproblems of the *89*-theory are the following:

Extension of Embe ings Problem

Given finite partial or ers P an Q P, oes every embe ing of P into D exten to an embe ing of Q into D?

Two natural subproblems of the *89*-theory are the following:

Extension of Embe ings Problem

Given finite partial or ers P an Q P, oes every embe ing of P into D exten to an embe ing of Q into D?

Lattice Embe ings Problem

Which finite lattices L can be embe e into D preserving not only partial or er but also join an meet)?

Two natural subproblems of the *89*-theory are the following:

Extension of Embe ings Problem

Given the ifficulty of the overall problem of eci ing the 89-theory of the enumeration egrees an of the $^0_2$-enumeration egrees, we are currently concentrating on the following subproblem of the Extension of Embe ings Problem for the $^0_2$-enumeration egrees:

1-Point Extensions of ntichains

Deci e, given a finite antichain $P = fa_0; :::; a_n g$ an 1-point extensions $Q_S = fa_0; :::; a_n; x_S g$ an $Q^T = fa_0; :::; a_n; x^T g$ for some *nonem ty* subsets $S; T$ $f0; :::; ng$ where $x_S < a_i$ i $i \; 2 \; S$; an $x^T > a_i$ i $i \; 2 \; T$), whether any embe ing of P can be exten e to an embe ing of Q_S for some such S or to an embe ing of Q^T for some such T not mapping the new element to $\mathbf{0}_e$ or $\mathbf{0}_e^0$)?

Given the ifficulty of the overall problem of eci ing the *89*-theory of the enumeration egrees an of the $^0_2$-enumeration egrees, we are currently concentrating on the following subproblem of the Extension of Embe ings Problem for the $^0_2$-enumeration egrees:

1-Point Extensions of ntichains

Deci e, given a finite antichain $P = fa_0;:::;a_ng$ an 1-point extensions $Q_S = fa_0;:::;a_n;x_Sg$ an $Q^T = fa_0;:::;a_n;x^Tg$ for some *nonem ty* subsets $S;T \quad f0;:::;ng$ where $x_S < a_i$ i $i\ 2\ S$; an $x^T > a_i$ i $i\ 2\ T$), whether any embe ing of P can be exten e to an embe ing of Q_S for some such S or to an embe ing of Q^T for some such T not mapping the new element to $\mathbf{0}_e$ or $\mathbf{0}_e^0$)?

It is always possible to exten an embe ing of a finite antichain P to an embe ing of the antichain $Q_; = Q^;$.)

The context for our subproblem is the two following earlier results:

Theorem hma 1989 cf. hma , Lachlan 1998))

1. There is an *Ahmad air* of $^0_2$-enumeration egrees (a; b), i.e., there are incomparable egrees a an b such that any egree v < a is b.

The context for our subproblem is the two following earlier results:

Theorem hma 1989 cf. hma , Lachlan 1998))

T

The context for our subproblem is the two following earlier results:

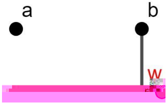
Theorem hma 1989 cf. hma , Lachlan 1998))

1. There is an *Ahmad air* of ${}^0_2$-enumeration egrees (a; b), i.e., there are incomparable egrees a an b such that any egree v < a is b.
2. There is no *ymmetri Ahmad air* of ${}^0_2$-enumeration egrees, i.e., there are no incomparable egrees a an b such that any egree v < a is b, an any egree w < b is a.

These are examples of *89*-statements blocking P Q_0 but not P $Q_0; Q_1$:

P

Q_0



Q_1

We can handle the case of Q_S:

Theorem in Progress (Goh, Lempp, Ng, M. Soskova)

Fix $n > 1$ and $\mathcal{S} \subseteq P(\{0, \dots, n\}) - \{\emptyset\}$.
Let $S_0 = \{i \le n \mid \{i\} \in \mathcal{S}\}$, and let $S_1 = \{0, \dots, n\} - S_0$.
Then some embedding of P into $\mathcal{D}_e(\le \mathbf{0}_e')$ cannot be extended to an embedding of $Q_{\mathcal{S}}$ for any $S \in \mathcal{S}$ iff

1. $S_0 = \emptyset$; or
2. $\bigcup \mathcal{S} \notin \{0, 1, \dots, n\}$; or
3. $S_1 \neq \emptyset$ and there is an *alignment* $\alpha : S_0 \to P(S_1) - \{\emptyset\}$, i.e., a function such that
   - for each $i \in S_0$, $\{i\} \cup \alpha(i) \notin \mathcal{S}$, and
   - for each $F \subseteq S_0$ with $|F| > 1$, we have $\bigcap \{\alpha(i) \mid i \in F\} \notin \mathcal{S}$.

We can han le the case of Q_S:

Theorem in Progress Goh, Lempp, Ng, M. Soskova)

Fix $n > 1$ an S $P(f0;:::;ng)$ $f;g$.
Let $S_0 = fi$ $n\ j\ fig\ 2\ Sg$, an let $S_1 = f0;:::;ng$ S_0.
Then some embe ing of P into $D_e($ $\mathbf{0}_e^{\prime})$ cannot be exten e to an embe ing of Q_S for any $S\ 2\ S$ i

1. $S_0 = ;$; or
2. $\bigcup S\ 6\ f0;1;:::;ng$; or
3. $S_1\ 6\ ;$ an there is an *a ignment* $: S_0\ !\ P(S_1)$ $f;g$, i.e., a function such that
   - for each $i\ 2\ S_0$, $fig\ [\ (i)\ 2\ S$, and
   - for each F S_0 with $jFj > 1$, we have $\bigcap f\ (i)\ j\ i\ 2\ Fg\ 2\ S$.

The proof exten s both results of hma an combines them with minimal pair techniques.

s for Q^T, we have to take into account the following

Theorem Kalimullin, Lempp, Ng, Yamaleev 2022)

There is no cupping hma pair, i.e., an hma pair $(a; b)$ with a [$b = \mathbf{0}_e^0$.

s for Q^T, we have to take into account the following

Theorem Kalimullin, Lempp, Ng, Yamaleev 2022)

There is no cupping hma pair, i.e., an hma pair (a; b) with a [b = $\mathbf{0}_e^0$.

We conjecture that this is the only a itional obstruction when consi ering extensions by points above an antichain:

Conjecture

Fix $n > 1$ an $S; T$ $P(f0; : : : ; ng)$ $f; g$.
Then some embe ing of P into $D_e($ $\mathbf{0}_e^0)$ cannot be exten e to an embe ing of Q_S for any S 2 S or of Q^T for any T 2 T i

- Q_S satisfies the con itions of the Theorem in Progress, an
- any T 2 T contains only one element, or contains two elements $i; j$ with j 2 (i) from the Theorem in Progress).

Thanks!